Ergodic Theory and Measured Group Theory
Lecture 10

Exercise. For any ctbl infinite grp Γ, the shift action $\Gamma \curvearrowright X^\Gamma$ with any Bernoulli measure, i.e. $\mu := \nu^\Gamma$, where ν is a prob. m. on X, is free μ-a.e. $\frac{1}{\pi}$

Solution-hint. If X is nonatomic, e.g. $[0,1]$ with Lebesgue, then two coordinates being $=$ is measure 0. If has atoms, fix $\gamma \in \Gamma$. Show that the action of γ is free a.e. by looking at the cosets $\delta \cdot \langle \gamma \rangle$ (to act by γ on the right). Cases: $\langle \gamma \rangle$ is infinite and $\langle \gamma \rangle$ is finite (hence $\Gamma / \langle \gamma \rangle$ is infinite).

Groups from the perspective of ergodic theorems. The action of an invertible transformation is the same as that of $\mathbb{Z}$. Thus, we have a ptwise ergodic thm for pmp actions of $\mathbb{Z}$:

Theorem. Let $\mathbb{Z} \overset{a}{\curvearrowright} (X, \mu)$ be a pmp action (i.e. $\forall \gamma \in \mathbb{Z}$, $\mu(A) = \mu(\gamma \cdot A)$ for all Borel $A \subseteq X$). $\forall f \in L^1(X, \mu)$,

$$\lim_{n \to \infty} \frac{\text{Average of } f \text{ over } I_n \cdot x}{|I_n|} = \mathbb{E}(f \mid \mathcal{B}_a)(x) \text{ a.e.}$$

where $I_n := \{0, 1, \ldots, n\}$.

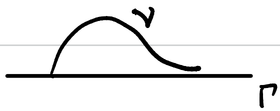
The property of the sequence (I_n) that makes the proof work is that $\forall \gamma \in \mathbb{Z}$, $\dfrac{|I_n \triangle \gamma \cdot I_n|}{|I_n|} \to 0$ as $n \to \infty$.

Def. For a ctbl gp Γ, a sequence (F_n) of finite subsets of Γ is called **Følner** if $\forall \gamma \in \Gamma$, $\dfrac{|F_n \triangle \gamma \cdot F_n|}{|F_n|} \to 0$.
The groups that admit such a sequence are called **amenable** ($\Leftrightarrow$ $\forall$ finite $S \subseteq \Gamma$, $\varepsilon > 0$, $\exists$ $F \subseteq \Gamma$ finite that is **(S,ε)-Følner**, i.e. $\forall \gamma \in S$ $\dfrac{|F \triangle \gamma \cdot F|}{|F|} < \varepsilon$).

Theorem. For a ctbl group Γ, TFAE:

(1) Γ is amenable, i.e. has a Følner sequence. 

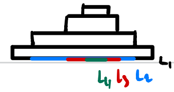
(2) Γ has **Reiter** functions, i.e. $\forall S \subseteq \Gamma$ finite, $\forall \varepsilon > 0$ $\exists$ finitely-supported prob. measure ν on Γ s.t. $\forall \gamma \in S$, $\|\nu - \gamma \cdot \nu\|_1 < \varepsilon$,



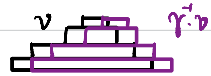
where $\gamma \cdot \nu(x) := \nu(\gamma^{-1} x)$.

(3) Γ admits a finitely-additive invariant prob. measure defined on all subset of Γ

(3') Γ admits a positive mean, i.e. $\exists$ a $\lambda \in \ell^\infty(\Gamma)^*$ s.t. $\forall f \neq 0$ & $f \geq 0$ in $\ell^\infty(\Gamma)$, $\lambda(f) > 0$.

Proof-sketch. (2) $\Rightarrow$ (1). Each $\nu \in P(\Gamma)$ finitely-supported admits a layered cake decomposition $\nu =$  $= \sum_{i=1}^{k} r_i \mathbb{1}_{L_i}$,

where $L_1 \supseteq L_2 \supseteq \dots \supseteq L_k$.



Claim. $\forall \gamma \in \Gamma$, $\forall x \in \Gamma$, $(\mathbb{1}_{L_i} - \mathbb{1}_{\gamma \cdot L_i})(x)$ are either 0 or have the same sign, i.e. $\nexists\, i \neq j$ s.t. the sign is different.

Proof. $L_1 \supseteq L_2 \supseteq \dots \supseteq L_k$ $\quad x:$ $\;1\;1\;1\;0\;0\;0\;0\;0 \dots 0$

$\gamma L_1 \supseteq \gamma L_2 \supseteq \dots \supseteq \gamma L_k$ $\quad\;\;\; 1\;1\;1\;1\;1\;0\;0\;0 \dots 0$

$\square$

$$\forall \gamma,\ |\nu - \gamma^{-1}\cdot\nu| = \left|\sum_{i=1}^{k} r_i \left(\mathbb{1}_{L_i} - \mathbb{1}_{\gamma L_i}\right)\right| = \sum_i r_i \left|\mathbb{1}_{L_i} - \mathbb{1}_{\gamma L_i}\right| = \sum_i r_i \mathbb{1}_{L_i \triangle \gamma L_i}$$

$$\sum_{\gamma \in S} \|\nu - \gamma^{-1}\cdot\nu\|_1 = \sum_{\gamma \in S} \sum_{i=1}^{k} r_i\, |L_i \triangle \gamma L_i|$$

$$(\text{Fubini}) \qquad = \sum_{i=1}^{k} r_i\, |L_i| \cdot \sum_{\gamma \in S} \frac{|L_i \triangle \gamma L_i|}{|L_i|}.$$

For a fix S, let ν be s.t. $\|\nu - \gamma^{-1}\nu\|_1 < \varepsilon/S$,

hence $\varepsilon > \sum_{\gamma \in S} \|\nu - \gamma^{-1}\nu\|_1 = \sum_{i=1}^{k} r_i |L_i| \sum_{\gamma \in S} \frac{|L_i \triangle \gamma L_i|}{|L_i|}$.

$1 = \|\nu\|_1 = \sum_{i=1}^{k} r_i\, |L_i|$, so is a convex combination.

Thus by PHP, $\exists i$ s.t. $\sum_{\gamma \in S} \frac{|L_i \triangle \gamma L_i|}{|L_i|} < \varepsilon$. $\square$

(1) ⇒ (2). trivial.

(2) ⇒ (3). Take a weak*-limit of finitely-supported measures.

(3') Or given a Følner sequence (F_n), define the measure λ on Γ by setting, for $A \subseteq \Gamma$,

$$\lambda(A) := \lim_{n \overset{*}{\to} \infty} \frac{|A \cap F_n|}{F_n}, \text{ where the limit}$$

is along an ultrafilter so it always exists.

By def, this gives an inv. fin. additive prob. measure.

(3) ⇒ (2). By geometric Hahn-Banach. $\boxed{\diagdown}$

<u>Examples and closure properties.</u>

- Finite groups are amenable.
- $\mathbb{Z}$ is amenable (intervals).
- Amenable groups are closed under amenable extensions, i.e. $\forall$ $1 \to \Gamma_1 \to \Gamma \to \Gamma_2 \to 1$ short exact sequence, if Γ_1 and Γ_2 are amenable, then so is Γ.
- In particular, amenable groups are closed under product.
- $\mathbb{Z}^d$ is amenable.
- Amenable groups are closed under ctbl increasing unions.
- $\bigoplus_{n \in \mathbb{N}} \mathbb{Z}^n$ is amenable. Also all abelian groups are amenable.
- Hence solvable groups are amenable.
- Closed under subgroups. Ex.
- Much more...

We said that the key ingredient in the proof of the ptwise ergodic theorem for $\mathbb{Z}$ was Følnerness of (I_n). So is the ptwise ergodic theorem true for all amenable groups along any Følner sequence? Yes (basically)!

Theorem (Lindenstrauss 2000). The ptwise ergodic thm holds along any tempered Følner sequence, i.e. $\forall$ amenable gp Γ & a tempered Følner sequence (F_n), any pmp action a of Γ on (X,μ), $\forall f \in L^1(X,\mu)$,

$$\lim_{n\to\infty} \frac{\text{Average of } f \text{ over } F_n \cdot x}{|F_n|} = \mathbb{E}(f \mid \mathcal{B}_a).$$

A Følner sequence (F_n) is called *tempered* if $\forall n$

$$\left|\left(\bigcup_{i<n} F_i\right)\cdot F_n\right| < C\cdot |F_n| \quad \text{for a constant } C.$$

Obs. Every Følner sequence has a tempered subsequence.

Proof. Let $S := \bigcup_{i<n} F_i$. $\square$

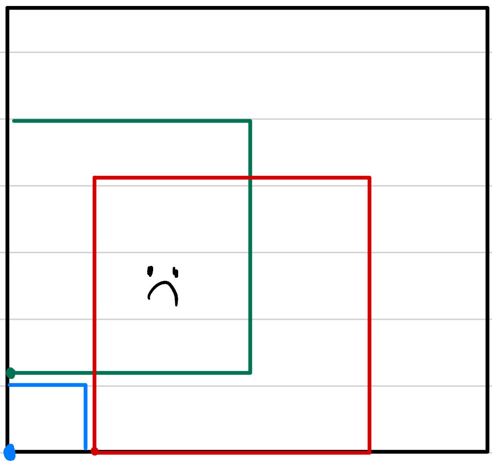
A stronger condition than tempered is the *Tempel'man condition*:

$$\left|\left(\bigcup_{i\leq n} F_i\right)\cdot F_n\right| < C\cdot |F_n|$$

For an increasing (F_n), this is the same as $|F_n^{-1} F_n| < C \cdot |F_n|$.

Tempel'man's theorem. Ptwise ergodic holds along increasing Tempelman Følner sequences. In particular, for boxes in $\mathbb{Z}^d$.

For boxes in $\mathbb{Z}^d$ the constant C is 2^d. Recalling the proof for $\mathbb{Z}$, it amounts to tiling arbitrarily well a large Følner set by smaller Følner sets placed at a point that likes it. Let's see how difficult it is to do in $\mathbb{Z}^2$:



Big Følner set